
PIECEWISE (BRANCH) FUNCTIONS

A shipping company might offer a discount if the shipping weight exceeds a certain minimum. For example, the shipping fee might be \$3/lb for a shipment weighing less than 20 lbs, and \$2/lb for a weight of 20 lb or more.



Our federal income tax system uses a progressive tax; higher-income earners pay a higher *percentage* of their taxable income than lower-income earners. For instance (and this is an oversimplification), those who earn under \$50,000/yr might pay at a rate of 12% of their income, those who earn between \$50,000 and \$90,000 pay 15%, and those who earn more than \$90,000 pay 20%.



EXAMPLE 1: For the shipping company in the Introduction,

- A. Find the cost to ship 12 lbs.
- B. Find the cost to ship 35 lbs.
- C. Express the shipping cost, S , as a branch function of the shipping weight, w .

Solution:

- A. Since $w = 12$, which is less than 20, it costs \$3/lb, for a total of $12 \text{ lbs} \times \$3/\text{lb} = \mathbf{\$36}$.
- B. 35 lbs at \$2/lb (because $35 \text{ lbs} > 20 \text{ lbs}$) comes to **\$70**.
- C. In English, we could say:
- If $w < 20$, then $S = 3w$.
- If $w \geq 20$, then $S = 2w$.

In our new notation, this is written

$$S = \begin{cases} 3w & \text{if } w < 20 \\ 2w & \text{if } w \geq 20 \end{cases}$$

EXAMPLE 2: Find the piecewise (branch) function for the income tax in the Introduction, and then use the function to find the tax on an income of \$88,000.

Solution: Here's the branch function, where I = income and T = tax:

$$T = \begin{cases} 12\% \times I & \text{if } I < 50,000 \\ 15\% \times I & \text{if } 50,000 \leq I \leq 90,000 \\ 20\% \times I & \text{if } I > 90,000 \end{cases}$$

To find the tax on an income of \$88,000, we find which branch of the tax function the number 88,000 lies in. It lies in the middle branch, so we use the formula $15\% \times I$, giving a tax of

$$T = 15\% \times I = 0.15(88,000) = \mathbf{\$13,200}$$

EXAMPLE 3: Let's look more closely at the absolute value function:

$$f(x) = |x|$$

We know, for example, that $f(9) = 9$, $f(-17) = 17$, and $f(0) = 0$.

Here's one way we can define **absolute value**, using the **if/then** form:

If $x \geq 0$, then $|x| = x$

If $x < 0$, then $|x| = -x$

As a branch function, we write

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

In programming, it might be written:

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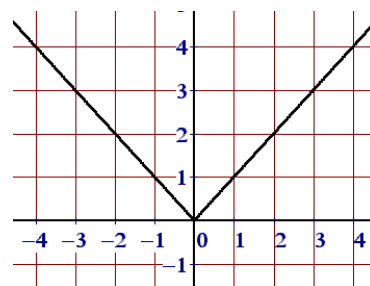
If  $x \geq 0$  Then
    Abs ( $x$ ) =  $x$ 
Else
    Abs ( $x$ ) =  $-x$ 
Endif

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In summary, we call a function described in the three examples above a **branch function** (or piecewise function), because the formula we use to calculate the output depends upon which branch of the domain the input lies in.

For example, in the *absolute value* branch function, suppose $x = 9$. Since $9 \geq 0$, we use the formula in the top branch; so $|9| = 9$. If we take x to be 0, we are again in the top branch, and thus $|0| = 0$. But if we take $x = -17$, then x falls in the bottom branch, whose formula is $-x$. So $|-17| = -(-17) = 17$, just as it should be.

Note: The definition of absolute value we've just written tells us that the domain of f is $\mathbb{R}$, all real numbers. Here's how you can tell: Choose any real number, and then see that it must fall into exactly one of the two "if's"; that is, the real number you chose is either ≥ 0 or it's < 0 . All the bases are covered, so the absolute value function is defined for all real numbers.



EXAMPLE 4: Consider the branch function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ 2x & \text{if } x > 3 \end{cases}$$

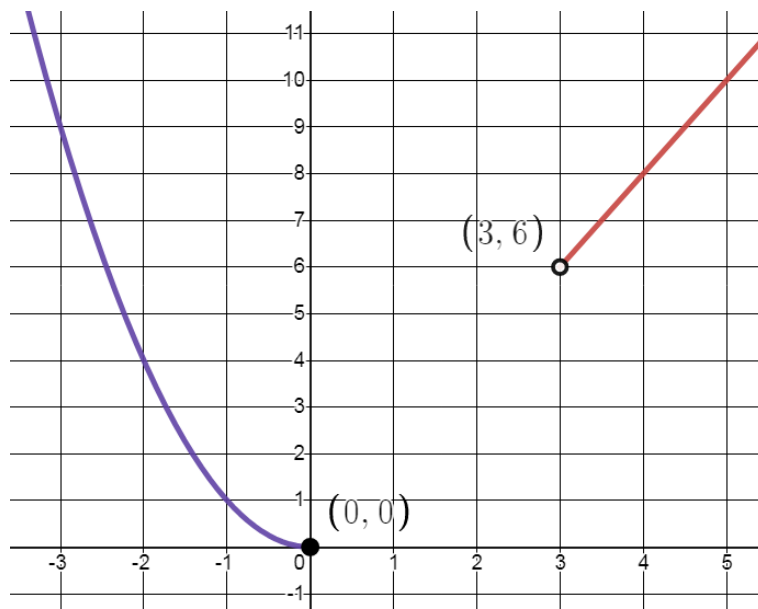
- I. Calculate: a. $f(-8)$ b. $f(0)$ c. $f(3)$
- d. $f(200)$ e. $f\left(\frac{\pi}{2}\right)$
- II. What is the domain of f ?
- III. Graph f .
- IV. Analyze all the interesting limits.

Solution: To calculate function values, we first need to determine whether the input we're looking at falls in the first branch ($x \leq 0$), in the second branch ($x > 3$), or possibly in no branch at all.

- I. a. Since $-8 \leq 0$, $f(-8) = 8^2 = \mathbf{64}$.
- b. Also, $0 \leq 0$, so $f(0) = 0^2 = \mathbf{0}$.
- c. The value $x = 3$ is not in either branch. Thus, $f(3)$ is **undefined**.
- d. Because $200 > 3$, $f(200) = 2(200) = \mathbf{400}$.
- e. Note that $\frac{\pi}{2}$ lies between 0 and 3 — it falls through the cracks of the domain. Therefore, $f\left(\frac{\pi}{2}\right)$ is **undefined**.

II. The domain

III. For $x \leq 0$, th



Note that there's a solid dot at the origin, because f is a parabola for any $x \leq 0$. In other

words, 0 is in the domain; thus the point (0, 0) is on the graph. On the other hand, the point (3, 6) is not on the graph, since 3 is not in the domain — hence the open dot.

IV. Now for some limits:

As $x \rightarrow 3$ (from the right), $f(x) \rightarrow 6$,

but x cannot approach 3 from the left.

As $x \rightarrow 0$ (from the left), $f(x) \rightarrow 0$,

yet x cannot approach 0 from the right.

As $x \rightarrow \infty$, $f(x) \rightarrow \infty$ (up the straight line).

As $x \rightarrow -\infty$, $f(x) \rightarrow \infty$ (up the parabola).

Final Comment: To say that $x \leq 0$ means that $x \in (-\infty, 0]$. Also, if $x > 3$, then $x \in (3, \infty)$. Thus, an equivalent form of the original branch function f in this example is

$$f(x) = \begin{cases} x^2 & \text{if } x \in (-\infty, 0] \\ 2x & \text{if } x \in (3, \infty) \end{cases}$$

The notation $x \in A$ means that x is an **element** of A .

Also, while I use the word “if,” some books use “for.”

Homework

1. Let h be the function defined by $h(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$
 - A. What is the domain of h ?
 - B. Find $h(13)$, $h(0)$, and $h(-\pi)$.
 - C. Graph h .

2. Consider the branch function defined by $f(x) = \begin{cases} x^2 & \text{for } x \leq 0 \\ 3x & \text{for } x > 2 \end{cases}$
 - A. Calculate: a. $f(-8)$ b. $f(0)$ c. $f(2)$

d. $f(200)$ e. $f(\pi/3)$

B. What is the domain of f ?

C. Graph f .

3. Let g be the piecewise function defined by

$$g(x) = \begin{cases} x^3 & \text{if } x \in (-\infty, -2) \\ -x^2 & \text{if } x \in [-1, 1] \\ 2 & \text{if } x \in (1, \infty) \end{cases}$$

A. Calculate: a. $g(1/2)$ b. $g(99)$ c. $g(-4)$ d. $g(-1.5)$
 e. $g(-5)$ f. $g(-2)$ g. $g(-1)$ h. $g(1)$

B. What is the domain of g ?

C. Graph g .

Check out:

<https://www.mathsisfun.com/sets/functions-piecewise.html>

❑ *TO ∞ AND BEYOND*

Define the function f by $f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \pi & \text{if } x \text{ is rational} \end{cases}$

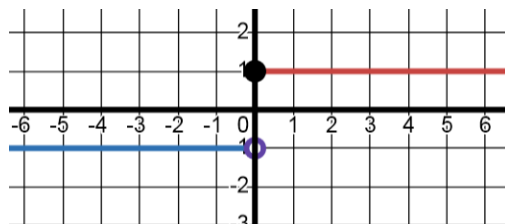
Find $(f \circ f)\left(\frac{2}{3}\right)$.

Solutions

1. A. The domain is $\mathbb{R}$, all real numbers, since every real number falls in exactly one of the two branches of the domain.

B. $1; 1; -1$

C.



2. A. a. 64 b. 0 c. Undefined d. 600 e. Undefined

B. $(-\infty, 0] \cup (2, \infty)$

C.



3. A. a. $-\frac{1}{4}$ b. 2 c. -64 d. Undefined
 e. -125 f. Undefined g. -1 h. -1

B. It's just the union of the three intervals given in the function:

$$(-\infty, -2) \cup [-1, 1] \cup (1, \infty)$$

But note that the last two intervals can be combined into $[-1, \infty)$.

C.

